Matching Chiral Perturbation Theory and the Dispersive Representation of the Scalar $K\pi$ Form Factor

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Abstract: We perform a matching of the two loop-chiral perturbation theory representation of the scalar $K\pi$ form factor to a dispersive one. Knowing the value of F_K/F_π and $f_+(0)$ in the Standard Model (SM) allows to determine two $O(p^6)$ LECs, the slope of the scalar form factor and the deviation of the Callan-Treiman theorem. Going beyond the SM and assuming the knowledge of the slope of the scalar form factor from experiment, the matching allows us to determine the ratio of F_K/F_π, $f_+(0)$, a certain combination of non-standard couplings, the deviation of the Callan-Treiman theorem and the two $O(p^6)$ LECs.

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1   Introduction

One privileged framework for studying meson and baryon properties in the low-energy domain is chiral perturbation theory (ChPT), the effective field theory of the Standard Model (SM). It is well known that it involves so-called low energy constants (LECs) which describe the influence of "heavy" degrees of freedom not contained explicitly in the Lagrangian. Determining these LECs is a difficult non-perturbative problem. It is, however, extremely important to pin them down in order to reach predictivity. Different attempts are made: phenomenological evaluation based on experimental information at low energies, resonance saturation, sum rules, resonance chiral theory, lattice QCD as well as matching [1]. Here we will be concerned with two QCD quantities, the pion and kaon decay constants, F_π and F_K respectively and two of the $O(p^6)$ LECs C_{12} and C_{34} [2]. These last two enter the calculation of two very important quantities, namely the strangeness changing vector and scalar form factors in ChPT at two loops. For example, the knowledge of the scalar form factor at the so-called Callan-Treiman (CT) point as well as the one of the vector form factor at zero momentum transfer enable one to test the SM [3, 4, 5]. There are thus many theoretical works related to the extraction of these quantities [5]-[14]. Also they are extensively investigated in the four experiments by NA48 [15, 16], KLOE [17, 18], KTEV [19] and ISTRA [20]. A determination of the two $O(p^6)$ LECs C_{12} and C_{34} has already been done for example in Refs. [21, 22] using some a priori experimental knowledge of the pion and kaon decay constants. Here we want to go somewhat further. It was realized in Ref. [23] that, independently of the problems related to quark mixing, the actual values of these two decay constants are known only if one assumes the electroweak couplings of the SM. We want to investigate some consequences of this observation. For this, we will use the dispersive representation of the $K_{\mu3}$ scalar form factor introduced in Ref. [5] and do a matching to the two-loop calculation of Bijnens and Talavera [10]. That is we will concentrate here on standard ChPT. Would the SU(3) quark condensate be much smaller than the SU(2) one as discussed in Refs. [24, 25, 26] would the results presented here be different. A study of this is beyond the scope of the letter. From the matching and assuming the SM, we will be able to determine the two $O(p^6)$ LECs, the slope of the scalar form factor and the deviation of the Callan-Treiman theorem. Going beyond the SM and assuming the knowledge of the slope of the scalar form factor from experiment, the matching will allow us to determine the ratio of F_K/F_π, $f_+(0)$, a certain combination of non-standard couplings, the deviation of the Callan-Treiman theorem and the two $O(p^6)$ LECs.

In section 2, we discuss the decay constants and the vector $K\pi$ form factor. We show that they are known only in the framework of the SM and we introduce their modification from effects beyond the SM. We write these modifications in terms of three parameters which describe the coupling of right-handed quarks to the W-boson as well as the modification of the left-handed ones [23]. We will see however that our discussion is more general. We recall in section 3.1 the dispersive representation of the scalar form factor introduced in Ref. [5] and in section 3.2 its expression in a two-loop ChPT calculation [9, 10]. We do the matching of these two representations in section 3.3 and discuss the results both in the SM and beyond in section 4.

2 Decay constants and vector form factor

Fundamental QCD quantities are the pion and kaon decay constants defined as

$$\langle 0 | A_\mu^a | M^b(p) \rangle = i \delta^{ab} F_M p_\mu \,, \tag{2.1}$$

with A_μ the axial current operator and M the pion or the kaon mass, respectively. Indeed $4\pi F_\pi$ for example is the scale beyond which ChPT is not applicable anymore and thus enters naturally any ChPT calculations. It is common to use in these calculations $F_\pi = 92.4$ MeV and $F_K/F_\pi = 1.22$. The value for F_π (F_K) comes from the (radiative) inclusive decay rates for $\pi(K) \to \mu\nu(\gamma)$ [27]. Taking their ratio leads to the value of F_K/F_π just given. However the knowledge of these quantities involves the axial EW couplings of quarks to the W-boson. In order to determine them, one thus has to know these couplings. At present the only well-known quantity is the vector coupling $\mathcal{V}_{eff}^{ud}$ of the u and d quarks to W. It is very accurately determined from $0^+ \to 0^+$ transitions in nuclei assuming conservation of the vector current. Its value has been very recently updated [28] and is one standard deviation larger than in Ref. [29] with an uncertainty one third smaller,

$$\mathcal{V}_{eff}^{ud} = 0.97418(26). \tag{2.2}$$

($\mathcal{V}_{eff}^{ud}$ is also determined from the measurement of the neutron life time or pionic decays [30] but with a much larger uncertainty). Note that though the numerical results of this letter would be slightly affected by a small change in $\mathcal{V}_{eff}^{ud}$, the conclusions would not be modified. Thus what can presently be given very precisely are the values of the pion and kaon decay constants in the SM where the axial and vector couplings are equal. Physics beyond the Standard Model can lead to a small difference between the axial and vector couplings leading to some small contributions from right-handed currents (RHCs). Such a scenario has been discussed in Ref. [23] where three small parameters $\epsilon_{ns}, \epsilon_s$ and δ enter naturally into an effective non-quite decoupling theory beyond the leading order (LO) [31]. The first two describe such couplings of RHCs to non-strange and strange quarks to W while the last one modifies the left-handed couplings. We refer to Refs. [5, 23] for a more thorough discussion of these quantities. Let us just write here the modification of the vector and axial couplings at next-to-leading order (NLO) of this effective theory:

$$
\begin{aligned}
|\mathcal{V}_{eff}^{ud}|^2 &= \cos^2\hat\theta \,, \\
|\mathcal{A}_{eff}^{ud}|^2 &= \cos^2\hat\theta \,(1 - 4\,\epsilon_{ns}) \,, \\
|\mathcal{V}_{eff}^{us}|^2 &= \sin^2\hat\theta \left(1 + 2\frac{\delta + \epsilon_{ns}}{\sin^2\hat\theta}\right)(1 + 2\,\epsilon_s - 2\,\epsilon_{ns}) \,, \\
|\mathcal{A}_{eff}^{us}|^2 &= \sin^2\hat\theta \left(1 + 2\frac{\delta + \epsilon_{ns}}{\sin^2\hat\theta}\right)(1 - 2\,\epsilon_s - 2\,\epsilon_{ns}) \,.
\end{aligned}
\tag{2.3}
$$

In these expressions and in the following, the hat on a quantity denotes that its value is determined from the measured semi-leptonic branching ratio assuming the SM electroweak couplings. We also introduced here the Cabibbo angle $\hat\theta$ neglecting in the SM the ub CKM matrix element as suggested by the measurement of $\mathcal{V}_{eff}^{ub}$. With these expressions, one gets:

$$|\mathcal{V}_{eff}^{ud}|^2 + |\mathcal{V}_{eff}^{us}|^2 \equiv 1 + \Delta_{\text{unitarity}} = 1 + 2(\delta + \epsilon_{ns}) + 2(\epsilon_s - \epsilon_{ns})\sin^2\hat\theta, \tag{2.4}$$

that is a small deviation from unitarity can occur for the vector effective couplings of the effective theory. Using the relations above one obtains for the pion and kaon decay constants

$$|F_\pi|^2 = \hat{F}_\pi^2 (1 + 4\epsilon_{ns})$$

$$\left(\frac{F_K}{F_\pi}\right)^2 = \left(\frac{\hat{F}_K}{\hat{F}_\pi}\right)^2 \frac{\sin^2\hat{\theta}}{\cos^2\hat{\theta}} \frac{|\mathcal{A}_{eff}^{ud}|^2}{|\mathcal{A}_{eff}^{us}|^2} = \left(\frac{\hat{F}_K}{\hat{F}_\pi}\right)^2 \frac{1 + 2(\epsilon_s - \epsilon_{ns})}{1 + \frac{2}{\sin^2\hat{\theta}}(\delta + \epsilon_{ns})} , \tag{2.5}$$

where

$$\hat{F}_\pi = (92.3 \pm 0.1) \text{ MeV} , \qquad \hat{F}_K/\hat{F}_\pi = 1.192 \pm 0.007 . \tag{2.6}$$

The value of $\hat{F}_K/\hat{F}_\pi$ is thus markedly smaller than what has been used so far in ChPT. It is obtained from the ratio $\Gamma_{K_{l2}^+(\gamma)}/\Gamma_{\pi_{l2}^+(\gamma)} = 1.3383(46)$ [4] of the inclusive decay rates for $\pi(K) \to \mu\nu$ and the value of $\mathcal{V}_{eff}^{ud}$ given in Eq. (2.2). The value of $\hat{F}_\pi$ is obtained from Refs. [32, 33, 34]

$$\sqrt{2}\hat{F}_\pi = \left(130.766 \left(\frac{0.9750}{\mathcal{V}_{eff}^{ud}}\right) + 0.156\, C_1\right) \text{ MeV} , \tag{2.7}$$

with $C_1 = -2.56 \pm 0.5$ [34].

Same discussion holds for the vector form factor. Its knowledge at zero momentum transfer is crucial for the determination of the CKM matrix element $\mathcal{V}_{eff}^{us}$. One has

$$|f_+^{K^0\pi^-}(0)|^2 = |\hat{f}_+^{K^0\pi^-}(0)|^2 \frac{\sin^2\hat{\theta}}{|\mathcal{V}_{eff}^{us}|^2} = \left[\hat{f}_+^{K^0\pi^-}(0)\right]^2 \frac{1 - 2(\epsilon_s - \epsilon_{ns})}{1 + \frac{2}{\sin^2\hat{\theta}}(\delta + \epsilon_{ns})} , \tag{2.8}$$

where the value obtained in the SM

$$\hat{f}_+^{K^0\pi^-}(0) = 0.9574(52) \tag{2.9}$$

comes from an average value of the K_{Le3} and K_{Se3} decay rate [3] leading to $|f_+(0)\mathcal{V}_{eff}^{us}| = 0.21615(55)$. Note that the same denominator enters both F_K/F_π and $f_+^{K^0\pi^-}(0)$ so that their ratio depends only on the difference $\epsilon_s - \epsilon_{ns}$. Also combining Eqs. (2.4) and (2.8), one sees that at NLO of the effective theory, the deviation from unitarity of the vector couplings can be related to the difference between the physical value of $f_+^{K^0\pi^-}(0)$ and its hat value. One has

$$\Delta_{\text{unitarity}} = \sin^2\hat{\theta} \left(\frac{|\hat{f}_+^{K^0\pi^-}(0)|^2}{|f_+^{K^0\pi^-}(0)|^2} - 1\right) . \tag{2.10}$$

Clearly this deviation can only be very small, its sign depending on the exact value of $f_+^{K^0\pi^-}(0)$. In fact, from the lattice results, one expects $-2.5 \times 10^{-3} < \Delta_{\text{unitarity}} < 8 \times 10^{-4}$.

It was discussed in Ref. [23] that the parameters ϵ_{ns} and δ should be small, less than a percent. Note however that in Eqs. (2.5) and (2.8) the quantity $\delta + \epsilon_{ns}$ is multiplied by the not so small quantity $1/\sin^2\hat{\theta}$, we will thus refrain in the following from expanding the denominator in these expressions. On the other hand, ϵ_s could be enhanced to a few percent level which could be explained for example by an inverted hierarchy in right-handed flavour mixing. One

expects from these estimates that F_K/F_π and $f_+(0)$ should be more affected than F_π by the presence of RHCs.

Our discussion will in fact be more general. Indeed, in the following, we will not consider any modification of F_π from its value obtained with the effective couplings of the SM. As just said these are expected to be rather small. Thus only two quantities will play a role in the following which can be chosen as

$$\alpha = \frac{1 + 2(\epsilon_s - \epsilon_{ns})}{1 + \frac{2}{\sin^2\theta}(\delta + \epsilon_{ns})} \quad \text{and} \quad \beta = \frac{1 - 2(\epsilon_s - \epsilon_{ns})}{1 + \frac{2}{\sin^2\theta}(\delta + \epsilon_{ns})} \ . \tag{2.11}$$

They just parametrize our ignorance of the physical values of F_K/F_π and $f_+(0)$ if there is physics beyond the SM. For the reader who prefers to think in terms of these quantities it is easy to rewrite $\epsilon_s - \epsilon_{ns}$ and $\delta + \epsilon_{ns}$ as a function of α and β.

3 Matching

3.1 Dispersive representation

A dispersive representation of the scalar form factor was introduced in Ref. [5]. It is based on a twice subtracted dispersion relation and reads:

$$\bar{f}_0(t) \equiv \frac{f_0^{K^0\pi^-}(t)}{f_0^{K^0\pi^-}(0)} = \exp\left[\frac{t}{\Delta_{K\pi}}(\ln C - G(t))\right] , \tag{3.1}$$

with

$$G(t) = \frac{\Delta_{K\pi}(\Delta_{K\pi} - t)}{\pi} \int_{(M_K+M_\pi)^2}^{\infty} \frac{ds}{s} \frac{\phi(s)}{(s - \Delta_{K\pi})(s - t - i\epsilon)} \ .$$

and $\phi(s)$ the phase of the form factor. It has many advantages. First, it introduces the value of the form factor at the Callan-Treiman point $\Delta_{K\pi} = M_K^2 - M_\pi^2$, a quantity C which is not affected by chiral corrections beyond $SU(2) \times SU(2)$. Thus these are of order $\mathcal{O}(m_u, m_d)$ while the slopes have larger corrections of the order of $\mathcal{O}(m_s)$. Second, it allows to test the Standard Model. Indeed one can relate the scalar form factor at the Callan-Treiman point to the quantity $\epsilon_s - \epsilon_{ns}$. One has:

$$C \equiv \bar{f}_0(\Delta_{K\pi}) = \frac{F_K}{F_\pi} \frac{1}{f_+^{K^0\pi^-}(0)} + \Delta_{CT} , \tag{3.2}$$

which using Eqs. (2.5) and (2.8), leads to

$$C = \frac{\hat{F}_K}{\hat{F}_\pi} \frac{1}{\hat{f}_+^{K^0\pi^-}(0)}(1 + 2(\epsilon_s - \epsilon_{ns})) + \Delta_{CT} = B_{exp}(1 + 2(\epsilon_s - \epsilon_{ns})) + \Delta_{CT} \ . \tag{3.3}$$

Hence one obtains from the values, Eqs. (2.6) and (2.9),

$$\ln C = 0.2188 \pm 0.0035 + \Delta\epsilon \tag{3.4}$$

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where $\Delta\epsilon \equiv \Delta_{CT}/B_{exp} + 2(\epsilon_s - \epsilon_{ns})$ and $B_{exp} = 1.2446 \pm 0.0041$. Expanding $\bar{f}_0(t)$

$$\bar{f}_0(t) = 1 + \lambda_0 \frac{t}{M_\pi^2} + \frac{1}{2}\lambda_0' \left(\frac{t}{M_\pi^2}\right)^2 + \cdots , \tag{3.5}$$

the linear slope is given in terms of $\ln C$ as

$$\lambda_0 = \frac{M_\pi^2}{\Delta_{K\pi}}(\ln C - G(0)) , \tag{3.6}$$

with $G(0) = 0.0398 \pm 0.0036 \pm 0.0020$ [5] whereas the curvature reads

$$\lambda_0' = \lambda_0^2 - 2\frac{M_\pi^4}{\Delta_{K\pi}}G'(0) = \lambda_0^2 + (4.16 \pm 0.50) \times 10^{-4}. \tag{3.7}$$

Note that in order to get a very precise description of $\bar{f}_0(t)$ over the entire physical region it is necessary to do an expansion up to third order [35]. Here we will concentrate on the region around $t = 0$.

3.2   ChPT to two loops

The scalar form factor was calculated to two loops in ChPT in Ref. [10]. These authors introduced the quantity

$$\tilde{f}_0(t) = f_+(t) + \frac{t}{M_K^2 - M_\pi^2}(f_-(t) + 1 - F_K/F_\pi) = f_0(t) + \frac{t}{M_K^2 - M_\pi^2}(1 - F_K/F_\pi) . \tag{3.8}$$

The main advantage is that this quantity has no dependence on the L_i^r at order p^4, only via order p^6 contributions. It, however, depends on the $\mathcal{O}(p^6)$ LECs C_i^r in the following way:

$$\begin{aligned}
\tilde{f}_0(t) &= 1 - \frac{8}{F_\pi^4}(C_{12}^r + C_{34}^r)(M_K^2 - M_\pi^2)^2 + 8\frac{t}{F_\pi^4}(2C_{12}^r + C_{34}^r)(M_K^2 + M_\pi^2) \\
&\quad - \frac{8}{F_\pi^4}t^2 C_{12}^r + \overline{\Delta}(t) + \Delta(0) .
\end{aligned} \tag{3.9}$$

The quantities $\overline{\Delta}(t)$ and $\Delta(0)$ have contributions from loops, thus depend on F_π, and from the LECs L_i. Note that L_5 is related to F_K/F_π. $\overline{\Delta}(t)$ and $\Delta(0)$ can in principle be calculated to order p^6 accuracy with the knowledge of the L_i^r to order p^4 accuracy. $\overline{\Delta}(t)$ has been parametrized in the physical region as:

$$\begin{aligned}
\overline{\Delta}(t) &= -0.25763t + 0.833045t^2 + 1.25252t^3 \quad [K_{e3}^0], \\
\overline{\Delta}(t) &= -0.260444t + 0.846124t^2 + 1.33025t^3 \quad [K_{e3}^+].
\end{aligned} \tag{3.10}$$

Different sets of L_i^r have been obtained from a fit to $K_{\ell 4}$ data to two loops [36]. The error from the values of the different sets of L_i^r is about 0.0013 at $t = 0.13$ GeV^2. Contributions from the loops and the L_i^r to $\Delta(0)$ are:

$$\Delta(0) = -0.0080 \pm 0.0057[\text{loops}] \pm 0.0028[L_i^r] , \tag{3.11}$$

where the central value arises from a cancellation between $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ terms $-0.008 = -0.02266 \, (p^4) + 0.01130 \, (p^6 \, \text{pure loops}) + 0.00332 \, (p^6 \, L_i)$. For more details, see Ref. [10].

3.3 Basic Formulae

Relating the dispersive representation to the two-loop ChPT calculation will allow us to determine the deviation from the Callan-Treiman theorem, F_K/F_π, the LECs C_{12} and C_{34} as well as either the slope of the form factor or the quantity $\delta + \epsilon_{ns}$ once one has fixed the quantities $\epsilon_s - \epsilon_{ns}$ and either $\delta + \epsilon_{ns}$ or the slope of the form factor, respectively. Taking the derivative of Eq. (3.9), the ChPT expression for the slope is:

$$\lambda_0 f_+(0) = \frac{M_\pi^2}{\Delta_{K\pi}} \left(\frac{F_K}{F_\pi} - 1 \right) + \frac{8M_\pi^2 \Sigma_{K\pi}}{F_\pi^4} (2C_{12} + C_{34}) + M_\pi^2 \overline{\Delta}'(0) \ , \tag{3.12}$$

with $\Sigma_{K\pi} = M_K^2 + M_\pi^2$. Combining the curvature obtained from Eq. (3.9),

$$\lambda_0' f_+(0) = -\frac{16M_\pi^4}{F_\pi^4} C_{12} + M_\pi^4 \overline{\Delta}''(0) \ , \tag{3.13}$$

with the two-loop result for $f_+(0)$

$$f_+(0) = 1 + \Delta(0) - \frac{8}{F_\pi^4}(C_{12} + C_{34})\Delta_{K\pi}^2 \ , \tag{3.14}$$

one gets an expression for $2C_{12} + C_{34}$. Inserting it into Eq. (3.12), using further the dispersive relation, Eq. (3.7) and expressing $f_+(0)$ and F_K/F_π in terms of the hat quantities, Eqs. (2.5) and (2.8), one obtains a second order equation for the slope λ_0 whose solution reads:

$$\lambda_0 = -\frac{M_\pi^2}{\Sigma_{K\pi}} \left(1 - \sqrt{1 - 2\frac{\Sigma_{K\pi}^2}{\Delta_{K\pi}} \left(\frac{Y}{\Delta_{K\pi}} - G'(0) \right)} \right) \tag{3.15}$$

with

$$Y = 1 - \frac{\Delta_{K\pi}}{\Sigma_{K\pi}} \frac{\hat{F}_K}{\hat{F}_\pi \hat{f}_+(0)} (1 + 2(\epsilon_s - \epsilon_{ns})) \tag{3.16}$$

$$- \frac{1}{\hat{f}_+(0)} \left(1 + \Delta(0) + \frac{\Delta_{K\pi}^2}{2}\overline{\Delta}''(0) - \frac{\Delta_{K\pi}}{\Sigma_{K\pi}} \left(1 - \Delta_{K\pi}\overline{\Delta}'(0) \right) \right) (1 + \epsilon_s - \epsilon_{ns})\sqrt{1 + y} \ .$$

Contrary to $\ln C$ which depends only on $\epsilon_s - \epsilon_{ns}$, λ_0 is a function of both quantities $\epsilon_s - \epsilon_{ns}$ and $y = 2(\delta + \epsilon_{ns})/\sin^2\hat{\theta}$. Once λ_0 is known, all the other quantities are determined in terms of $\epsilon_s - \epsilon_{ns}$ and y. F_K/F_π, $f_+(0)$ are given by Eqs. (2.5) and (2.8) respectively and

$$C_{12} = \frac{F_\pi^4}{16} \left(-\frac{\lambda_0' f_+(0)}{M_\pi^4} + \overline{\Delta}''(0) \right) \ , \tag{3.17}$$

$$C_{34} = \frac{F_\pi^4}{8\Delta_{K\pi}^2} (1 + \Delta(0) - f_+(0)) - C_{12} \ .$$

One has trivially from Eqs. (3.2) and (3.6)

$$\Delta_{CT} = B_{exp} \left(\frac{\Delta_{K\pi}}{M_\pi^2}\lambda_0 + G(0) - \ln B_{exp} - 2(\epsilon_s - \epsilon_{ns}) \right) \ . \tag{3.18}$$

$\Delta(0)$	$\epsilon_s - \epsilon_{ns}$	$\Delta_{\text{unitarity}}$ (10^{-3})	λ_0 (10^{-3})	Δ_{CT} (10^{-2})	$f_+(0)$	F_K/F_π	C_{12} (10^{-6})	C_{34} (10^{-6})
-0.008	SM	SM	15.20	-0.118	0.957 *	1.192 *	-0.421	6.480
	0	-1.5	15.03	-0.368	0.972	1.210	-0.484	3.971
	0	-3.1	14.85	-0.622	0.987	1.229	-0.550	1.344
	0	1.5	15.37	0.127	0.943	1.174	-0.362	8.879
	0	3.1	15.53	0.369	0.930	1.157	-0.306	11.176
-0.0165	SM	SM	14.46	-1.193	0.957 *	1.192 *	-0.170	4.741
	0	-1.5	14.30	-1.428	0.972	1.210	-0.235	2.235
0.0005	SM	SM	15.93	0.948	0.957 *	1.192 *	-0.683	8.229
	0	-1.5	15.75	0.684	0.972	1.210	-0.743	5.718

Table 1: Values of the slope of the form factor λ_0, the deviation from the Callan-Treiman theorem Δ_{CT}, the value of the vector form factor at zero momentum transfer $f_+(0)$, the ratio of the pion and kaon decay constants F_K/F_π, two $\mathcal{O}(p^6)$ LECs C_{12} and C_{34} as a function of the non standard couplings $\epsilon_s - \epsilon_{ns}$ to the W-boson and the deviation from unitarity $\Delta_{\text{unitarity}}$ of the effective couplings. The star means that the quantities are known from experiment, Eqs. (2.6) and (2.9). The dependence on the ChPT input quantity $\Delta(0)$ is also shown.

4   Results and Conclusion

We will not try here to get exact results but more trends of what can be expected from such a matching. Indeed, in order to do the matching, one has to use values for $\Delta(0)$ and $\overline{\Delta}(t)$ which have been determined using $F_\pi = 92.4$ MeV and $F_K/F_\pi = 1.22$. Thus our results will not be completely consistent since we will in the following determine F_K/F_π from Eq. (2.5). Also if $\epsilon_{ns} \neq 0$, F_π will be modified, see Eq. (2.5). However, we do not expect much changes in the result would one do a consistent calculation. Indeed in $\Delta(0)$ the contribution from the L_i is rather small and a small uncertainty was found in $\overline{\Delta}(t)$ while using different sets of L_i's, see also Ref. [37]. Besides, as already mentioned one expects values of ϵ_{ns} smaller than a percent so that F_π^2 would be changed by at most 4%. All these effects can, to our opinion, very well be accounted by the rather conservative uncertainties given for $\Delta(0)$, Eq. (3.11). We will thus vary $\Delta(0)$ within its error bars to see how the results are affected. Ultimately, we would of course like to study the dependence of the results on F_π since it would enable one to determine independently δ and ϵ_{ns}. It would indeed be very interesting to test the quark-lepton universality which implies $\delta = 0$ [38]. However the conservative uncertainty on $\Delta(0)$, Eq. (3.11), is too big, as we will see, to really get very precise results. Note also that since the fits were done in Ref. [36], new $K_{\ell 4}$ data are available. New fits should certainly be performed [37] leading to an updated value for $\Delta(0)$.

In the following, we will be using the central value for $\Delta(0) = -0.008$, for $\overline{\Delta}(t)$ the values from the fit to neutral kaons and $2M_\pi^4 G'(0)/\Delta_{K\pi} = -4.66 \times 10^{-4}$. We will also consider the deviation from unitarity of the vector effective couplings, $\Delta_{\text{unitarity}}$, Eq. (2.4) instead of the quantity $\delta + \epsilon_{ns}$. It is easy to recover the values of this quantity from Eq. (2.4) if needed. We will consider two different scenarios. In the first one, we will fix $\epsilon_s - \epsilon_{ns} = 0$ and study the dependence of the results on $\Delta_{\text{unitarity}}$. In the second one, we will study the case $\epsilon_s - \epsilon_{ns} \neq 0$.

• First, we will assume that we are in the SM. In that case, $\delta = \epsilon_{ns} = \epsilon_s = 0$. The results are given in table 1. F_K/F_π and $f_+(0)$ are the hat quantities determined from experiments as discussed in section 2, see Eqs. (2.6) and (2.9). With the updated value of $\mathcal{V}_{eff}^{ud}$, they are now in good agreement with the recent lattice results for $F_K/F_\pi = 1.189(7)$ [39] and $f_+(0) = 0.9609(51)$ [40] obtained with staggered and DWF fermions respectively. Note however that the value of F_K/F_π from Ref. [39] is somewhat on the lower side of most of the lattice results. A rather small value for F_K/F_π has been obtained recently from the CP-PACS/JLQCD collaboration, however most of the SU(3) lattice results give central values around 1.21, see Refs. [41, 42]. Lattice values for $f_+(0)$ are $0.95 < f_+(0) < 0.98$ [41, 43] while the widely used quark model of Leutwyler and Roos [44] gives $f_+(0) = 0.961 \pm 0.008$. λ_0 is on the large side of the experimental results while consistent with the KLOE result as obtained from a linear parametrization for the scalar form factor and a quadratic one for the vector [18]. It has however recently been understood that the use of a linear parametrization is not appropriate. It leads to a value for the slope of the scalar form factor larger than it actually is [45]. Δ_{CT} is very small as expected from the NLO result in ChPT in the isospin limit [6]

$$\Delta_{CT}^{NLO} = (-3.5 \pm 8) \cdot 10^{-3} \tag{4.1}$$

where the error is a conservative estimate assuming some typical corrections of $\mathcal{O}(m_{u,d})$ and $\mathcal{O}(m_s)$ [46]. The LEC C_{12} is found to be negative. Resonance exchange models give negative values of the order of 10^{-5} for a scalar mass exchange of $M_S \sim 980$ MeV which corresponds to the a_0. Other masses have also been considered [22]. Taking M_S between 1 GeV and 1.5 GeV one gets $-9 \cdot 10^{-6} \lesssim C_{12} \lesssim -1.8 \cdot 10^{-6}$. Assuming that the LECs determined within these resonance exchange models correspond to a scale equal to M_S and evolving them to the ρ scale one gets values between $-7.8 \cdot 10^{-6}$ and $4.0 \cdot 10^{-6}$ for the range of the scalar masses discussed above [21]. In that reference, $C_{12} = (0.3 \pm 5.4) \cdot 10^{-7}$ for a value of $\lambda_0 = 0.0157 \pm 0.0010$ where the central value corresponds to $f_+(0) = 0.976$. This is consistent with our findings within the error bars. However they have a smaller result for the sum $(C_{12} + C_{34})(M_\rho) = (3.2 \pm 1.5) \cdot 10^{-6}$. Thus calculating the C_i's contribution to $f_+(0)$

$$f_+(0) = -\frac{8}{F_\pi^4}(C_{12} + C_{34})(M_K^2 - M_\pi^2)^2, \tag{4.2}$$

our result is twice as large in absolute value than the one given in that letter or in the pioneering work [44], $f_+(0) = -0.016 \pm 0.008$. In the case of Δ_{CT}, the C_i's contribution is given by:

$$\Delta_{CT}|_{C_i} = \frac{16}{F_\pi^4}(2C_{12} + C_{34})M_\pi^2(M_K^2 - M_\pi^2). \tag{4.3}$$

Subtracting it to the value of Δ_{CT} given in the table, one finds $\Delta_{CT} - \Delta_{CT}|_{C_i} = -6.68 \cdot 10^{-3}$ in very good agreement with the two loop contribution recently evaluated in Ref. [11], as it should. Note that adding to the expansion, Eq. (3.5), the t^3 term from Eq. (3.10), one obtains a good parametrization of Eq. (3.1) up to the Callan-Treiman point.

• Giving a small value to $\delta + \epsilon_{ns}$ while keeping $\epsilon_s - \epsilon_{ns} = 0$, that is breaking the unitarity of the vector couplings, Eq. (2.4) by a small amount, the value for λ_0 given in the second entry in table 1 is consistent with the one obtained in Ref. [12] and calculated along the line of a dispersion theoretical approach of Ref. [47]. In this framework where, differently from the one discussed here, a two channel approach has been used and only one subtraction is performed,

$\Delta(0)$	$\epsilon_s - \epsilon_{ns}$	λ_0 (10^{-3})	$\Delta_{\text{unitarity}}$ (10^{-3})	Δ_{CT} (10^{-2})	$f_+(0)$	F_K/F_π	C_{12} (10^{-6})	C_{34} (10^{-6})
-0.008	-0.005	14.00	-2.804	-0.623	0.984	1.213	-0.234	1.534
	-0.032	9.01	-3.148	-1.178	0.987	1.152	1.107	-0.216
-0.0165	-0.0012	13.99	-2.416	-1.579	0.980	1.218	-0.202	0.666
	-0.028	9.00	-2.760	-2.130	0.983	1.157	1.132	-1.092
0.0005	-0.0088	14.00	-3.191	0.325	0.988	1.209	-0.264	2.400
	-0.0358	9.01	-3.535	-0.234	0.991	1.148	1.084	0.659

Table 2: Values of the deviation from unitarity $\Delta_{\text{unitarity}}$, the deviation from the Callan-Treiman theorem Δ_{CT}, the value of the vector form factor at zero momentum transfer $f_+(0)$, the ratio of the pion and kaon decay constants F_K/F_π, two $\mathcal{O}(p^6)$ LECs C_{12} and C_{34} as a function of $\epsilon_s - \epsilon_{ns}$ and the slope of the form factor where $\epsilon_s - \epsilon_{ns}$ is fixed from the measurement of $\Delta\epsilon$ as explained in the text. The dependence on the ChPT input quantity $\Delta(0)$ is also shown.

one needs two external input parameters. These authors use the value of the form factor at zero momentum and its value at the CT point. With $f_+(0) = 0.972(12)$ and $F_K/F_\pi = 1.203(16)$, they get $\lambda_0 = 0.0147(4)$. Varying $f_+(0)$ within the bounds from the lattice results one obtains $0.0148 \lesssim \lambda_0 \lesssim 0.0154$. As one decreases $\Delta_{\text{unitarity}}$, the values of λ_0, Δ_{CT} and the two $\mathcal{O}(p^6)$ LECs C_{12} and C_{34} decrease while the ones of $f_+(0)$ and F_K/F_π increase. One observes a strong dependence of Δ_{CT} and C_{34} on $\Delta_{\text{unitarity}}$ or equivalently, see Eq. (2.10), on $f_+(0)$. In the expression of C_{12}, Eq. (3.17), the first term on the right-hand side is negative and the second is positive. It turns out that both terms are of the same order of magnitude so that the sign and the value of C_{12} result from a delicate balance between the two terms. Here we have kept $\overline{\Delta}''(0)$ fixed from the fit to neutral kaons, first line Eq. (3.10), so that the different values obtained in the tables for C_{12} are only due to the changes in λ_0 and $f_+(0)$. Using for $\overline{\Delta}''(0)$ the value obtained in the fit to charged kaons would lead to a small change in the results. For example with this value one gets $C_{12} = -0.322 \cdot 10^{-6}$ in the SM, first line in table 1. Concerning C_{34} the first term is very sensitive to the difference between $1 + \Delta(0)$ and $f_+(0)$ which leads to the large observed variations in its values.

• In order to get smaller values of λ_0 as demanded by the central values of the NA48 and KTEV experiments as well as the KLOE one [18] when analyzed with the dispersive representation discussed in section 3.1, one must allow for $\epsilon_s - \epsilon_{ns} \neq 0$. Let us first assume the NA48 result [16] which is 5 σ deviation away from the SM one. The strategy here will be to reproduce the measured slope $\lambda_0 = (8.88 \pm 1.24) \times 10^{-3}$ from the dispersive analysis as well as the measured deviation from the Callan-Treiman theorem $\Delta\epsilon = -0.075 \pm 0.014$, Eq. (3.4). This leads to a negative value of $\epsilon_s - \epsilon_{ns}$ of the order of a few percent while $\delta + \epsilon_{ns}$ has to be extremely small and positive. As illustration, we show the results for $\lambda_0 = 9.0 \times 10^{-3}$ in table 2. This leads to values for F_K/F_π and $f_+(0)$ respectively, on the lower side of, somewhat larger than the lattice results. $f_+(0)$ is now much larger than in Ref. [44] but in agreement with Ref. [22]. Δ_{CT} turns out to be larger in absolute value than the NLO ChPT result, Eq. (4.1), however, it is within the expected uncertainty from higher orders. It leads to $\Delta\epsilon = -0.073$. Interestingly the LEC C_{12} is now much larger and positive. On the contrary, C_{34} becomes much smaller as one goes from the standard case to the NA48 result. Subtracting again the C_i's contribution, Eq. (4.3), to Δ_{CT} one now obtains a value twice as large as the quoted two loop results of
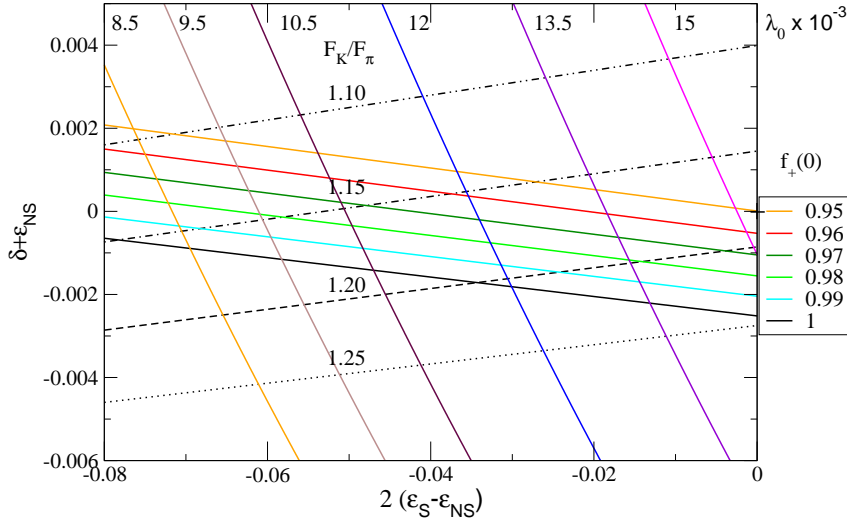
Figure 1: Lines of constant values for F_K/F_π, $f_+(0)$ as in Ref. [23] and λ_0 in the plane $\delta + \epsilon_{ns}$ and $2(\epsilon_s - \epsilon_{ns})$. λ_0 is calculated with the central value of $\Delta(0)$. Error on this quantity is larger than the one on F_K/F_π and $f_+(0)$, see discussion in the text.

Ref. [11] due to the smaller value of F_K/F_π. In the first entry of table 2, we give the result corresponding to the recent determination of the slope of the form factor by KLOE [18] using the dispersive parametrization. One can easily calculate what is their experimental value of $\Delta\epsilon$, using Eqs. (3.6) and (3.4). This leads to $\Delta\epsilon = -0.015 \pm 0.025$. The C_i's contribution to $f_+(0)$ and Δ_{CT} is respectively -0.0074 and 0.0010.

In both tables, we give results for larger and smaller values of $\Delta(0)$, corresponding to what is the dominant uncertainty in Eq. (3.15). For comparison, in table 1, we use the same values of $\epsilon_s - \epsilon_{ns}$ and $\delta + \epsilon_{ns}$ in all cases so that F_K/F_π and $f_+(0)$ are the same when varying $\Delta(0)$. The change in its value leads to a rather large shift in λ_0, Δ_{CT}, C_{12} and C_{34}. Thus the conservative uncertainty on the value of $\Delta(0)$ is unfortunately too big to really enable one to pin down these quantities with a very good precision. As can be seen, the matching together with all the experimental results on the slope of the scalar form factor available today fix the sign of $\epsilon_s - \epsilon_{ns}$ to be negative. With the effective couplings of the SM, λ_0 varies between 14.3×10^{-3} and 16.0×10^{-3}, that is the dependence with $\Delta(0)$ is large but can never afford such a small value as reported by the NA48 experiment. In table 2, we choose to keep λ_0 and $\Delta\epsilon$ approximately fixed. The NA48 and KLOE results from the dispersive analysis lead to values for $f_+(0) \sim 0.98$ in agreement with Ref. [22] while F_K/F_π is rather small in the NA48 case. Let us mention here that with such a small value of F_K/F_π the value of $\Delta(0)$ to be used should be closer to -0.0165 than to -0.008. Indeed the contribution of L_5 to $f_+(0)$ is positive [48]. One has in the case of the neutral kaons

$$f_+(0) = f_+(0)|_{\text{without} L_5} - 0.4136 L_5 + 5715.11 L_5^2, \qquad (4.4)$$

where the coefficient of L_5^2 is $-8\left(M_K^2 - M_\pi^2\right)^2 / F_\pi^4$, i.e. the same as the one of $C_{12} + C_{34}$, Eq. (4.2). A smaller value of F_K/F_π corresponds to a smaller value of L_5 and thus of $\Delta(0)$.

Note that varying $G'(0)$ within its error bar induces also a certain shift in the results essentially for Δ_{CT} and C_{12}.

In order to illustrate the results, we reproduce in figure 1 the one shown in Ref. [23] adding to the dependence of F_K/F_π and $f_+(0)$ on $\epsilon_s - \epsilon_{ns}$ and $\delta + \epsilon_{ns}$ the one of λ_0 using the central value of $\Delta(0)$. Note that while the errors on F_K/F_π and $f_+(0)$, which are purely experimental, are tiny, the ones on λ_0 coming from the two-loop ChPT calculations and not shown here are, as just discussed, rather large. However, as can be seen from the figure, a very precise knowledge of these three quantities would allow to pin down the existence of physics beyond the SM.

As we have seen, the actual status of experiments and lattice results does not, at present, exclude the presence of physics beyond the SM in terms of RHCs. As illustrated by the NA48 result, it could very well be that F_K/F_π and $f_+(0)$ is smaller, respectively larger than thought. Interestingly this would lead to completely different values of the two $O(p^6)$ LECs C_{12} and C_{34}. Since these enter other processes than the one discussed here their study might help clarifying the situation. Clearly more work is needed on the lattice side as well as on the ChPT side to reach the needed accuracy.

5   Acknowledgements

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